

Neutral 3-body system in a strong magnetic field: factorization and exact solutions

Yu.A.Simonov,

Institute of Theoretical and Experimental Physics
117118, Moscow, B.Chermushkinskaya 25, Russia

Abstract

Neutral systems containing two identical particles, in homogeneous magnetic field are shown to obey exact factorizable solutions both in nonrelativistic and relativistic formalism, similarly to the neutral two-body systems. Concrete examples of the helium atom and the neutron as a (ddu) system are considered.

1

The problem of composite system in the magnetic field has been always an important topic of investigation and of textbooks. One particular, and probably the simplest problem is that of the neutral two-particle system in MF, and a necessary step in its solution is the factorization of the c.m. and internal motion, which in the external MF is not a simple procedure. This problem was solved in [1, 2, 3, 4] in the nonrelativistic framework. Moreover, in [4] a general theorem was given, stating existence of a set of pseudomomenta in MF in a neutral N -body system. The factorization problem in the relativistic context was recently solved in [5], where the relativistic Hamiltonian for two-body neutral system was derived from the QCD path integral and applied to find the neutral meson spectrum in strong MF.

The idea of strong MF in our surroundings attracts nowadays a lot of researchers and finds support and confirmation in many areas. In astrophysics MF were known for a long time and very strong MF, up to 10^{18}

Gauss, were found in magnetars [6], very strong MF are possible in peripheral heavy-ion collisions at RHIC and LHC [7], and in early Universe [8].

On theoretical side the important development in last years was in the study of hydrogen atom in strong MF [9, 10, 11], where it was shown [9, 10], that the spectrum of hydrogen atom is stabilized in the limit of high MF due to e^+e^- loop corrections to the Coulomb force.

The case of neutral three-body system in MF is not less important, than its two-body analog; however here the development is less active. In particular, the general theorem of factorization of c.m. and internal motion found in the case of two-body neutral system, is not known for the three-body case, and the property of stabilization is not yet found.

The purpose of the paper is twofold. First, we present in section 2 the exact procedure of factorization in MF and demonstrate the nonrelativistic Hamiltonian for neutral three-body problem with two identical particles.

In section 3 this Hamiltonian is considered for the 3He and 4He atoms and some properties of the spectra are discussed. At the end of section 3 the relativistic generalization of the same Hamiltonian is derived and the neutron or Δ^0 isobar system is considered as physical examples, and dynamics and some properties of spectra are discussed. In section 5 the results are summarized and perspectives are given.

2 Three-body nonrelativistic Hamiltonian in the magnetic field

The nonrelativistic (Pauli) Hamiltonian for three particles with masses and charges $m_i, e_i, i = 1, 2, 3$ in the magnetic field \mathbf{B} has the standard form

$$H = \sum_{i=1}^3 \frac{(p_k^{(i)} - e_i A_k)^2 - e_i \boldsymbol{\sigma}^{(i)} \mathbf{B}}{2m_i} \equiv H_0 + H_\sigma \quad (1)$$

A general problem of few-body treatment in the magnetic field is the separation of the c.m. and relative (internal) motion. This problem is non-trivial and allows a factorizable solution for neutral two-body system after a special phase factor is introduced and conserved pseudomomenta are defined [1, 2, 3, 4, 5]. In [5] this solution was generalized to the relativistic two-body case. Below we show that the three-body system can be solved (factorized)

in the same way in one special case: when two of three particles are identical, i.e. $e_1 = e_2, m_1 = m_2$, but m_3 is arbitrary and $e_3 = -2e_1$.

We define $e_1 = e_2 = -\frac{e}{2}, e_3 = e, m_1 = m_2 = m$, and introduce Jacobi coordinates

$$\begin{cases} R_k = \frac{1}{m_+} \sum m_i z_k^{(i)}, \\ \eta_k = \frac{z_k^{(2)} - z_k^{(1)}}{\sqrt{2}}, \\ \xi_k = \sqrt{\frac{m_3}{2m_+}} (z_k^{(1)} + z_k^{(2)} - 2z_k^{(3)}). \end{cases} \quad (2)$$

where $m_+ = 2m + m_3$. Denoting

$$\mathcal{P}_k \equiv \frac{\partial}{i\partial R_k}, \quad q_k \equiv \frac{\partial}{i\partial \xi_k}, \quad \pi_k \equiv \frac{\partial}{i\partial \eta_k} \quad (3)$$

one has

$$p_k^{(i)} = \alpha_i \mathcal{P}_k + \beta_i q_k + \gamma_i \pi_k, \quad (4)$$

$$p_k^{(1)} = \frac{m}{m_+} \mathcal{P}_k + \sqrt{\frac{m_3}{2m_+}} q_k - \frac{1}{\sqrt{2}} \pi_k \quad (5)$$

$$p_k^{(2)} = \frac{m}{m_+} \mathcal{P}_k + \sqrt{\frac{m_3}{2m_+}} q_k + \frac{1}{\sqrt{2}} \pi_k \quad (6)$$

$$p_k^{(3)} = \frac{m_3}{m_+} \mathcal{P}_k - \sqrt{\frac{2m_3}{m_+}} q_k \quad (7)$$

In terms of P_k, q_k, π_k the Hamiltonian has the form

$$\begin{aligned} H_0 &= \frac{1}{2m} \left[\frac{m}{m_+} \mathbf{P} + \sqrt{\frac{m_3}{2m_+}} \mathbf{q} - \frac{\boldsymbol{\pi}}{\sqrt{2}} + \frac{e}{4} \left(\mathbf{B} \times \left(\mathbf{R} + \sqrt{\frac{m_3}{2m_+}} \boldsymbol{\xi} - \frac{\boldsymbol{\eta}}{\sqrt{2}} \right) \right) \right]^2 + \\ &+ \frac{1}{2m} \left[\frac{m}{m_+} \mathbf{P} + \sqrt{\frac{m_3}{2m_+}} \mathbf{q} + \frac{\boldsymbol{\pi}}{\sqrt{2}} + \frac{e}{4} \left(\mathbf{B} \times \left(\mathbf{R} + \sqrt{\frac{m_3}{2m_+}} \boldsymbol{\xi} + \frac{\boldsymbol{\eta}}{\sqrt{2}} \right) \right) \right]^2 + \\ &+ \frac{1}{2m_3} \left[\frac{m_3}{m_+} \mathbf{P} - \sqrt{\frac{2m_3}{m_+}} \mathbf{q} - \frac{e}{2} \left(\mathbf{B} \times \left(\mathbf{R} - \sqrt{\frac{2m^2}{m_+ m_3}} \boldsymbol{\xi} \right) \right) \right]^2 \\ &\equiv \frac{1}{2m} \left((\mathbf{J}^{(1)})^2 + (\mathbf{J}^{(2)})^2 \right) + \frac{1}{2m_3} \mathbf{J}^{(3)}. \end{aligned} \quad (8)$$

We now do the same step as in the two-body case and introduce the phase factor, which in our case has the form

$$\Psi(\mathbf{R}, \boldsymbol{\xi}, \boldsymbol{\eta}) = e^{-i\frac{\varepsilon}{4}(\mathbf{B} \times \mathbf{R}) \cdot \boldsymbol{\xi}} \sqrt{\frac{2m_+}{m_3}} e^{i\mathbf{P} \cdot \mathbf{R}} \varphi(\boldsymbol{\xi}, \boldsymbol{\eta}) \equiv e^{i\Gamma} \varphi \quad (9)$$

Acting with operators $\mathbf{J}^{(i)}$ on Ψ in the form of (9), one obtains a remarkable simplification,

$$H_0 \Psi = H_0 e^{i\Gamma} \varphi = e^{i\Gamma} \tilde{H}_0 \varphi, \quad (10)$$

where e.g. for $\mathbf{P} = 0$ one has

$$\begin{aligned} \tilde{H}_0 = & -\frac{1}{2m}(\Delta_{\boldsymbol{\xi}} + \Delta_{\boldsymbol{\eta}}) + \frac{1}{2m} \left(\frac{eB}{4} \right)^2 \left(\frac{m_+^2}{m_3^2} (\boldsymbol{\xi}_{\perp})^2 + (\boldsymbol{\eta}_{\perp})^2 \right) + \\ & + \frac{eB_k}{4m} \left(\frac{m_3 - 2m}{m_3} L_k^{(\boldsymbol{\xi})} + L_k^{(\boldsymbol{\eta})} \right) \end{aligned} \quad (11)$$

Here $L_k^{(\boldsymbol{\xi})}, L_k^{(\boldsymbol{\eta})}$ are Jacobi angular momenta

$$\mathbf{L}^{(\boldsymbol{\xi})} = \left(\boldsymbol{\xi} \times \frac{\partial}{i\partial \boldsymbol{\xi}} \right), \quad \mathbf{L}^{(\boldsymbol{\eta})} = \left(\boldsymbol{\eta} \times \frac{\partial}{i\partial \boldsymbol{\eta}} \right) \quad (12)$$

One can see in (11), that eigenfunctions of \tilde{H}_0 factorize,

$$\varphi(\boldsymbol{\xi}, \boldsymbol{\eta}) = f_1(\boldsymbol{\xi}_{\perp}) f_2(\boldsymbol{\eta}_{\perp}) \exp(ik_{\xi} \xi_3) + ik_{\eta} \eta_3 \quad (13)$$

where f_1, f_2 have standard form, e.g.

$$f_1(\boldsymbol{\eta}_{\perp}) = \frac{e^{il_{\xi} \varphi_{\xi}}}{\sqrt{2\pi}} \chi_n(x), \quad l_{\xi} = 0, \pm 1, \dots \quad (14)$$

$$\chi_n(x) = C_n e^{-\frac{x}{2}} x^{\frac{|l_{\xi}|}{2}} F(-n_{\xi}, |l_{\xi}| + 1, x), \quad (15)$$

where $n_{\xi} = 0, 1, 2, \dots, C_n$ is the normalization constant, and

$$x = \frac{eBm_+}{4m_3} \boldsymbol{\xi}_{\perp}^2,$$

while the corresponding energy is

$$E_{n_{\xi}} = \frac{k_{\xi}^2}{2m} + \frac{eBm_+}{2mm_3} \left(n_{\xi} + \frac{1 + |l_{\xi}|}{2} + \frac{m_3 - 2m}{2m_3} l_{\xi} \right). \quad (16)$$

In the same way for $f_2(\boldsymbol{\eta})$ one obtains

$$f_2(\boldsymbol{\eta}_\perp) \equiv \frac{\exp(il_\eta \varphi_\eta)}{\sqrt{2\pi}} \chi_{n_\eta}(y), \quad l_\eta = 0, \pm 1, \dots \quad (17)$$

$$\chi_{n_\eta}(y) = C_{n_\eta} e^{-\frac{y}{2}} y^{|l_\eta|/2} F(-n_\eta, |l_\eta| + 1, y), \quad (18)$$

where $y = \frac{eB}{4} \boldsymbol{\eta}_\perp^2$, $n_\eta = 0, 1, 2, \dots$ and the energy of the η -motion is

$$E_{n_\eta} = \frac{k_\eta^2}{2m} + \frac{eB}{2m} \left(n_\eta + \frac{1 + |l_\eta| + l_\eta}{2} \right) \quad (19)$$

The total energy of $(H_0 + H_\sigma)$ is

$$E = E_{n_\xi} + E_{n_\eta} - \sum_{i=1}^3 \frac{e_i \boldsymbol{\sigma}^{(i)} \mathbf{B}}{2m_i} \quad (20)$$

3 Physical examples

a) The case of Helium atom.

As a first example we consider the neutral atomic system of a helium atom, where two electrons play the role of identical particles, $m_1 = m_2 = m_e$, $e_1 = e_2 = -e$, $e_3 = 2e$, and m_3 is the mass of the helium nucleus, $m_3 = M(^3\text{He})$ or $m_3 = M(^4\text{He})$. In the first case for the ground state atom the electrons on the S level have opposite spin directions, and H_σ in (1) reduces to the magnetic moment term of ^3He nucleus, $H_\sigma = -\boldsymbol{\mu}(^3\text{He}) \mathbf{B}$. In the case of ^4He the term H_σ in (1) for the ground state is identically zero.

The Coulomb interaction

$$V_{\text{Coul}}(|\mathbf{z}_i - \mathbf{z}_j|) = -\frac{\alpha}{|\mathbf{z}_i - \mathbf{z}_j|},$$

is introduced in the standard way, adding to $H_0 + H_\sigma$ in (1) the term

$$V_{\text{Coul}}^{He}(\xi, \boldsymbol{\eta}) = 2V_{\text{Coul}} \left(\left| \sqrt{\frac{m_+}{2m_3}} \boldsymbol{\xi} - \frac{\boldsymbol{\eta}}{\sqrt{2}} \right| \right) + 2V_{\text{Coul}} \left(\left| \sqrt{\frac{m_+}{2m_3}} \boldsymbol{\xi} + \frac{\boldsymbol{\eta}}{\sqrt{2}} \right| \right) - V_{\text{Coul}}(|\sqrt{2}\boldsymbol{\eta}|). \quad (21)$$

In absence of MF, the Hamiltonian for the Helium atom is usually written in the form, which neglects finite nucleus mass corrections, namely

$$h = -\frac{1}{2m}(\Delta_1 + \Delta_2) - 2\alpha \left(\frac{1}{r_1} + \frac{1}{r_2} \right) + \frac{\alpha}{r_{12}}; \quad \mathbf{r}_i = \mathbf{z}_i - \mathbf{z}_3, \quad i = 1, 2, \quad (22)$$

while our Hamiltonian (11) for $\mathbf{P} = 0$ contains those (cf connection of $\boldsymbol{\eta}, \boldsymbol{\xi}$ and \mathbf{r}_i in (2)).

$$H = -\frac{1}{2m}(\Delta_{\xi} + \Delta_{\eta}) + V_{\text{Coul}}^{He}(\boldsymbol{\xi}, \boldsymbol{\eta}) = h - \frac{1}{2m_3} \left(\frac{\partial}{\partial \mathbf{r}_1} + \frac{\partial}{\partial \mathbf{r}_2} \right)^2. \quad (23)$$

Accurate calculations with the Hamiltonian (22) were being done for a long time [12] and have achieved an extremely high level of accuracy, see e.g. [13](see [14] for a recent review).

When strong MF is present, one can make the adiabatic approximation, as in the hydrogen atom case

$$H_{\text{adiab}} = -\frac{1}{2m_3} \left(\frac{\partial^2}{\partial \xi_3^2} + \frac{\partial^2}{\partial \eta_3^2} \right) + V_{\text{adiab}}(\xi_3, \eta_3), \quad (24)$$

where V_{adiab} is

$$V_{\text{adiab}}(\xi_3, \eta_3) = \int V_{\text{Coul}}^{He}(\boldsymbol{\xi}, \boldsymbol{\eta}) d^2\xi_{\perp} d^2\eta_{\perp} f_1^2(\xi_{\perp}) f_2^2(\eta_{\perp}) \quad (25)$$

As a result, the problem reduces to the one-dimensional three-body problem with Coulomb-like interaction. Neglecting c.m. corrections and the repulsive ee interaction term, one can factorize the wave function

$$\psi(\xi, \eta) \rightarrow \psi(r_1)\psi(r_2)$$

and in the adiabatic approximation one has a product of one-dimensional hydrogen-like functions, which obey stabilized dynamics at large MF [9, 10, 11]. This approach can be generalized in the same way, as it is done for the Helium atom without MF [12, 13, 14]. Since the ee interaction is repulsive one can establish a lower bound for the Helium binding energy in MF as the twice the limiting binding energy of the hydrogen atom in MF, i.e. $2 \cdot 1.74 \text{ keV} = 3.48 \text{ keV}$.

b) Neutral baryon case.

For neutron or Δ^0 baryon with the structure (ddu), of Ξ^0 baryon, (ssu), one can use the same strategy, as in the nonrelativistic case, but one must replace the starting form (1) by its relativistic analog (see [5] and refs. therein for details). The simple replacement holds in the external magnetic field, when one can keep, as in (1), the (2×2) structure, neglecting connection to the Dirac underground.

In this case the new Hamiltonian reads

$$H_{\text{rel}} = (H_0 + H_\sigma)(m_i \rightarrow \omega_i) + \frac{m_1^2 + \omega_i^2}{2\omega_i} + W, \quad (26)$$

where W contains gluon exchange (color Coulomb) V_{GE} , confinement V_{conf} , and spin-dependent and selfenergy terms.

The eigenvalue of (26), $M(\omega_1, \omega_2, \omega_3)$, is subject to the stationary point conditions $\frac{\partial M}{\partial \omega_i} \Big|_{\omega_i = \omega_i^{(0)}} = 0$, which define $\omega_i^{(0)}$ and the final eigenvalue $M(\omega_1^{(0)}, \omega_2^{(0)}, \omega_3^{(0)})$.

As in the nonrelativistic case, one finds the full separability of the Hamiltonian H_{rel} in case, when $\omega_1^{(0)} = \omega_2^{(0)} = \omega^{(0)}$. This is possible when not only $e_1 = e_2, m_1 = m_2$, but also spin projections of both quarks are equal, $\boldsymbol{\sigma}_1 \mathbf{B} = \boldsymbol{\sigma}_2 \mathbf{B}$. For this configuration one can calculate baryon masses as functions of B , as it was done in [5] for mesons.

We leave this topic for another publication.

4 Summary and conclusions

We have found a factorizable and fully separable form of nonrelativistic and relativistic Hamiltonian for a neutral three-body problem in MF with two identical particles.

We have found in this case exact solutions, and consider the physical examples of helium atom and neutral baryon. We demonstrate, that the Coulomb or gluon exchange attraction at large MF can cause the problem of stability as in the case of hydrogen atom, and for the helium atom the stability is ensured by that of the hydrogen atom. Our formalism may pave the road for the accurate calculations of three body system in MF. The author is grateful to M.A.Andreichikov and B.O.Kerbikov for discussions.

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